The Propagation Puzzle: Unraveling Community Clusters in the Stochastic Block Model Network

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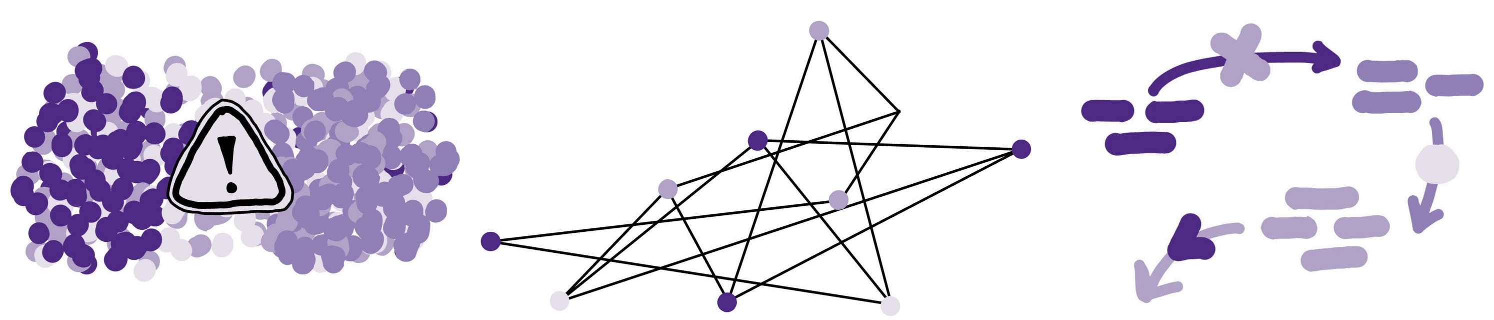
Searching for Clusters

Clustering (also known as **community detection**) is a classic problem in machine learning and network science that seeks to **identify clusters of related nodes in a network**



Community detection has critical applications in epidemiology (disease hotspots), finance (fraud detection), sociology (influence groups), and more.

LPA Algorithm

The **Label Propagation Algorithm (LPA)** is a popular clustering algorithm for many real-world applications due to its speed and scalability.

Despite its popularity, there is surprisingly little literature published on the behavior of LPA on the **stochastic block model (SBM)**

Goal: Provide empirical and theoretical characterizations of the LPA on the 2-community SBM

Graph Model

- Networks modeled as **unweighted, undirected graphs**
- Erdős-Rényi: one community (N, p)
- SBM: 2+ communities (N, p, q)

Algorithm

- The LPA is a way of identifying communities by propagating "opinions" through a network:
  1. Randomly assign labels 0 through N-1 to each node
  2. For each node, relabel it with the majority label of its neighbors, breaking ties towards the smaller label
  3. Repeat step 2 until **convergence** (no more changing labels) or termination

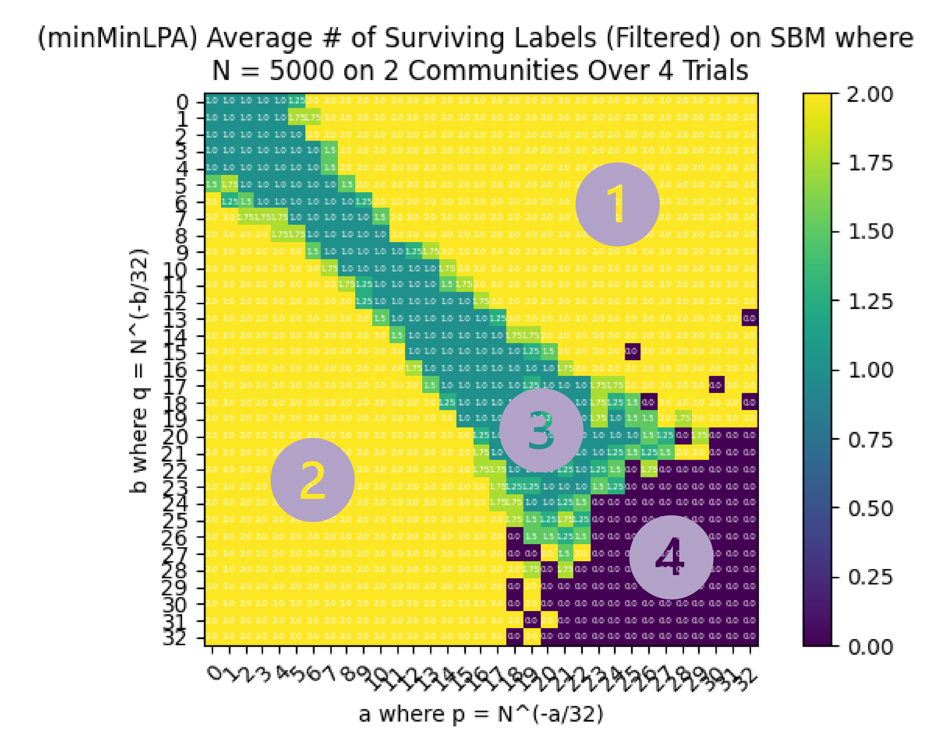
Empirical Simulations

Parameters

- **N** - # of nodes in network
- **numComm** - # of communities
- **p** – probability that any two nodes in the same community are connected
- **q** – probability that any two nodes in different communities are connected
- **cap** - # of iterations allowed before termination

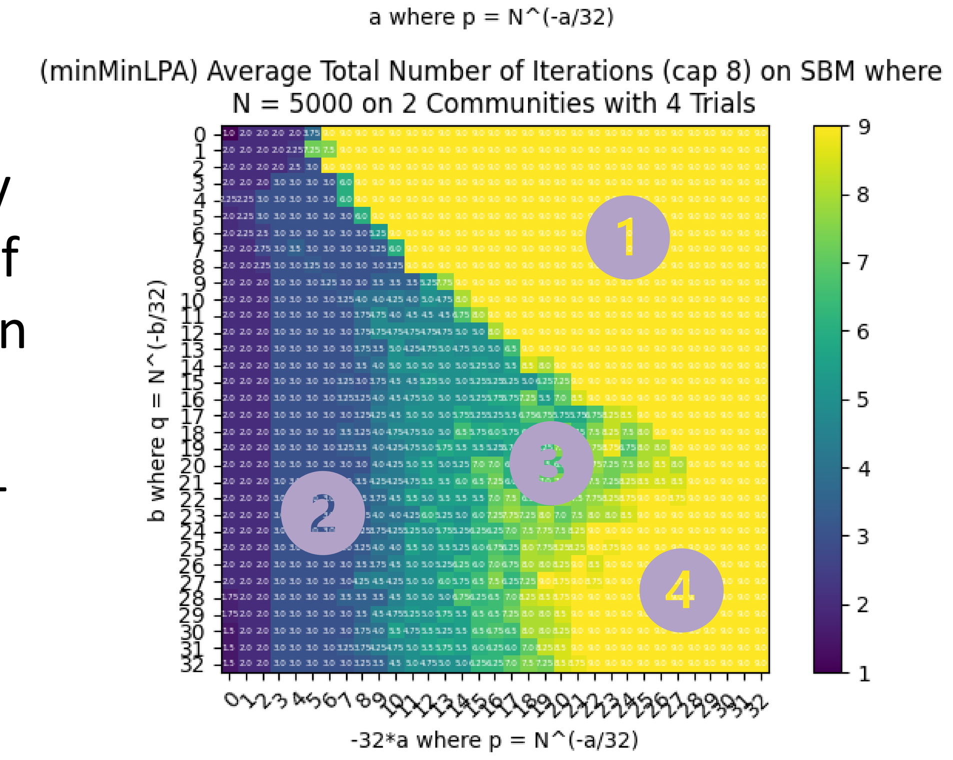
Behavioral Partition

Identified four distinct regions of the p-q parameter space that partition the space by convergence behavior

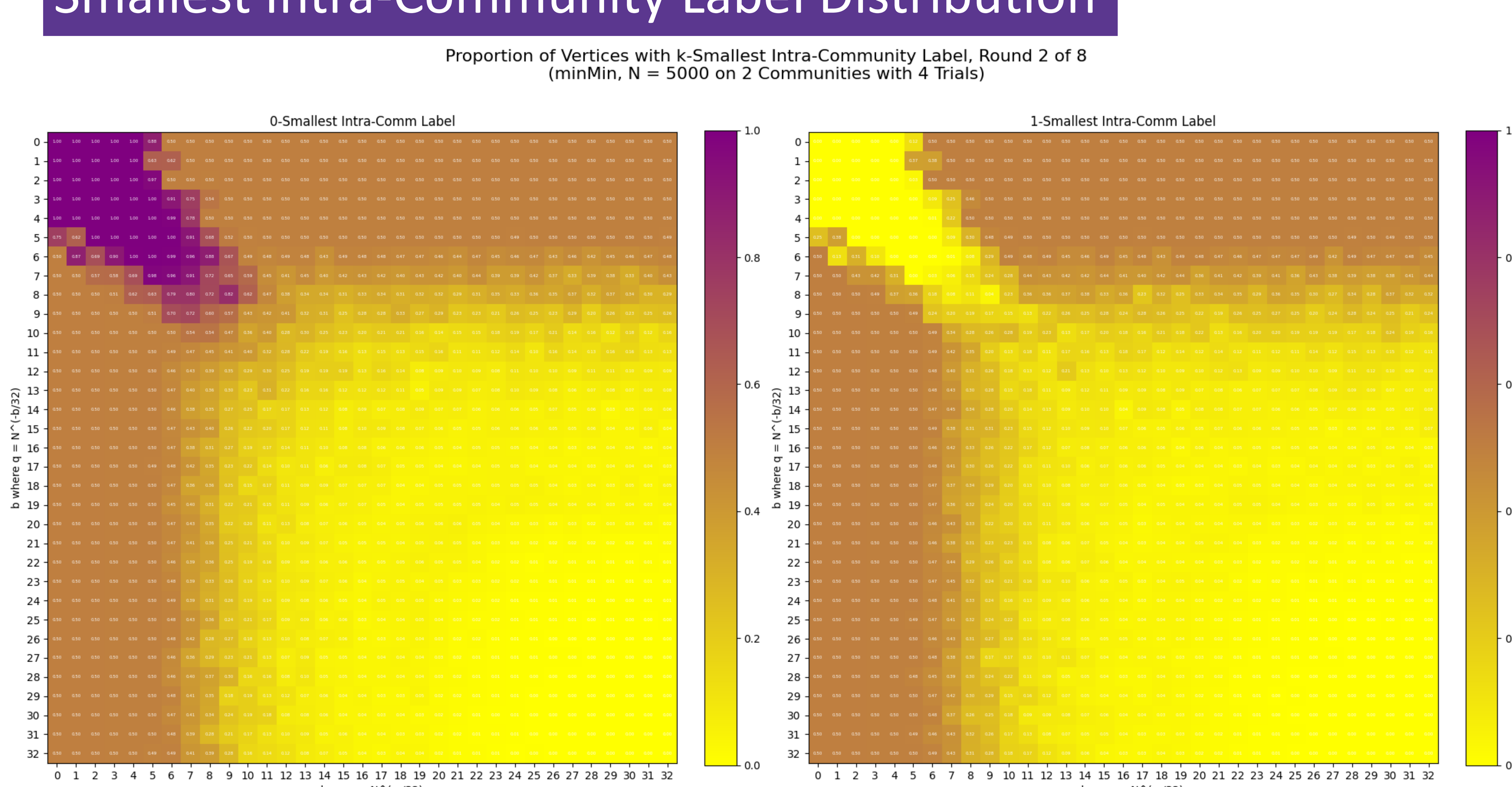


Switching

Despite the seemingly symmetric behavior of regions 1 and 2, region 2 exhibits stable convergence; region 1 exhibits periodic switching of labels



Smallest Intra-Community Label Distribution



Mathematical Proofs

Theorem 1 (Stable Convergence). *Let $G \sim SSBM(n,p,q)$ be a symmetric stochastic block model. All vertices are labeled as the smallest label of its community within 2 rounds of MinLPA w.h.p. if (i) $p \gg q$ and $p = \Omega(n^{-\frac{1}{4}+\epsilon})$ (an improvement from the current result [2], or (ii) $q \gg p$ and $q = \Omega(n^{-\frac{1}{4}+\epsilon})$.*

On the other side, not all vertices are labeled as the smallest label of its community within 2 rounds of MinLPA w.h.p. if $\max(p,q) = O(n^{-\frac{1}{4}})$.

Lemma 1 (Cross-Community Superiority). *Let $G \sim SSBM(n,p,q)$ with uniform random label initialization. Assume $p = \alpha n^{-a}$ and $q = \beta n^{-b}$. Assume that $v_{(1)} \in V_1$ and $v_{(2)} \in V_2$. Then $\forall v \in V_1$,*

$$Y_1(v) > Y_2(v)$$

where $Y_i(v)$ denotes the number of vertices in the neighborhood of v with label i after the first round. Also, $\forall v \in V_2$,

$$Y_2(v) > Y_1(v)$$

Lemma 2 (Local Superiority). *Let $G \sim SBM(n,p,q)$. Assume that $v_{(1)} \in V_1$ and $v_{(2)} \in V_2$. Then $\forall v \in V_1$,*

$$Y_1(v) > Y_i(v), \forall i \neq 1 \text{ with } v_{(i)} \in V_1$$

where $Y_i(v)$ denotes the number of vertices in the neighborhood of v with label i after the first round. Also, $\forall v \in V_2$,

$$Y_2(v) > Y_i(v), \forall i \neq 2 \text{ with } v_{(i)} \in V_2$$

Theoretical analysis proved tighter convergence conditions for accurate identification in two rounds; guarantee asymptotically accurate community detection for $SSBM(n,p,q)$ at the $\max\{p,q\} > n^{-1/4}$ threshold

Conclusions and Future Work

We have identified several critical behaviors (guaranteed convergence, guaranteed non-convergence, switching) of the LPA on the symmetric SBM in 2 communities, and have improved upon existing theoretical bounds on convergence criteria in two rounds. Future work may examine theoretical convergence in three or more rounds, precise behavior of the "Erdős-Rényi strip", LPA behavior over three or more communities, non-symmetric SBM, or different tiebreaking variation of the LPA.

References

[1] Kiwi, Marcos, Lyuben Lichev, Dieter Mitsche, and Paweł Prałat. "Label propagation on binomial random graphs." arXiv preprint arXiv:2302.03569 (2023).

[2] Kishore Kothapalli, Sriram V Pemmaraju, and Vivek Sardeshmukh. On the analysis of a label propagation algorithm for community detection. In International Conference on Distributed Computing and Networking, pages 255–269. Springer, 2013.