Signed Opinion Dynamics on Networks: Opposing vs. Repelling, Information, and Balance

Part 3: This time, it's personal

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MMSS Thesis Final Presentation
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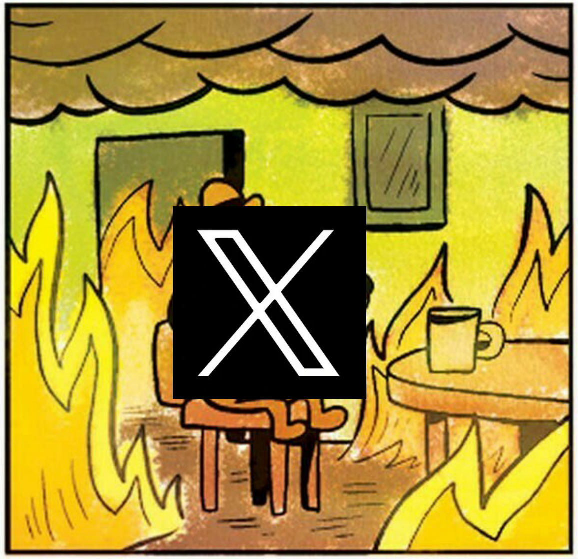
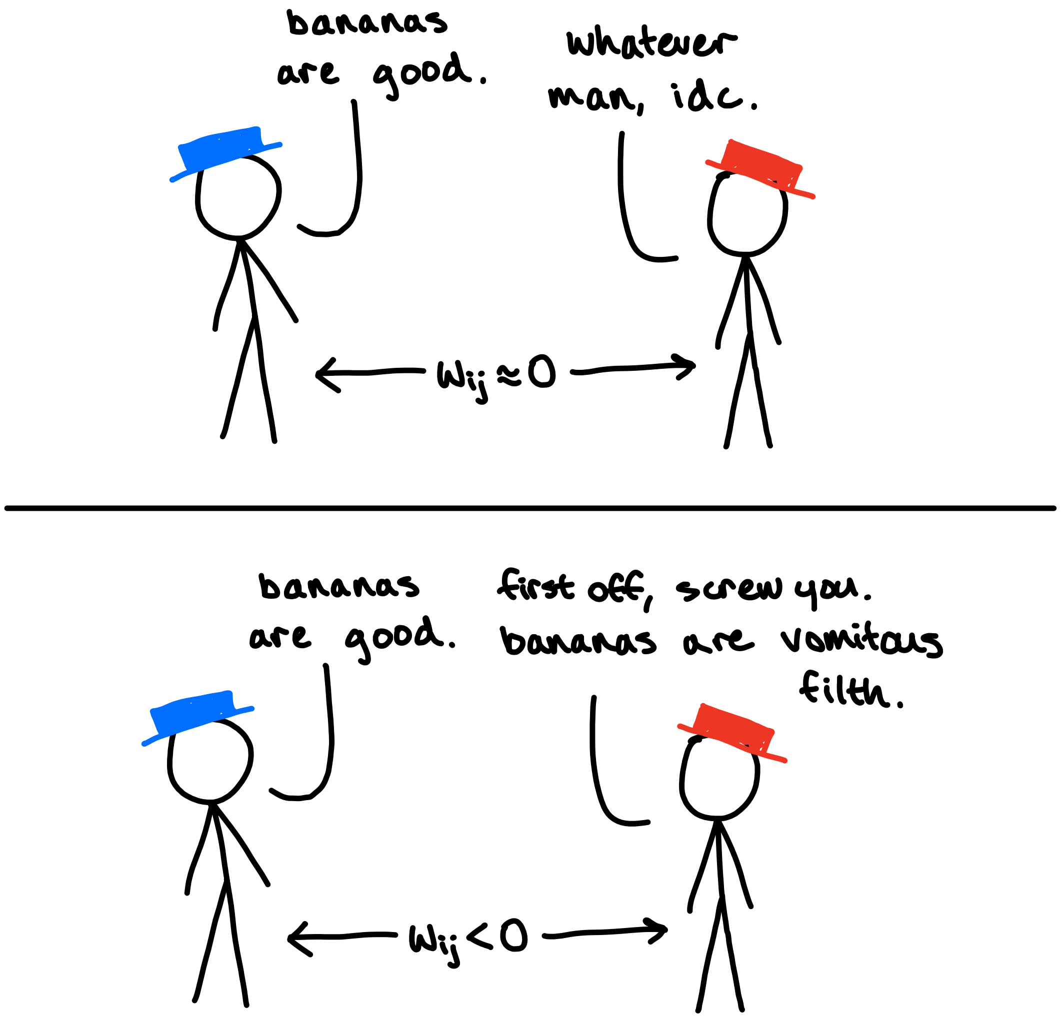
May 21, 2026

People Don't Like Their Opps





Background: Antagonism

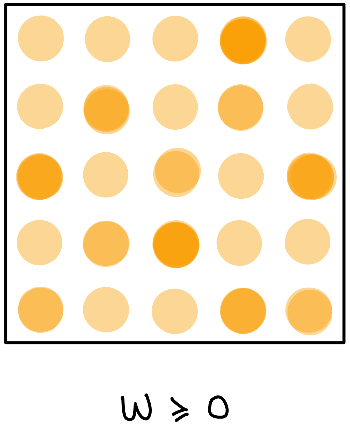
- In "DeGroot-type" models, you only have friends:
  - Update rule:

$$\mu_i(t+1) = \sum_{i\text{'s friends } j} w_{ij}\mu_j(t)$$

  - Matrix form:

$$\mu(t+1) = W\mu(t)$$

(where W row-stochastic, nonnegative)

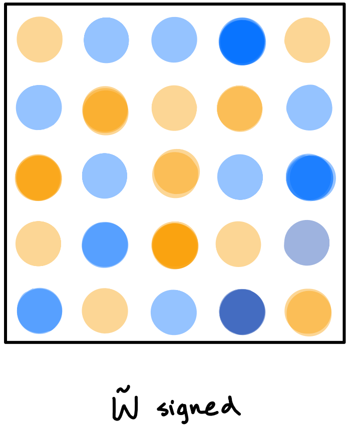


- In antagonistic models, you also have opps:
  - Opposing rule:

$$\mu_i(t+1) = \sum_{i\text{'s friends } j} w_{ij}\mu_j(t) + \sum_{i\text{'s opps } j} (-1)w_{ij}\mu_j(t)$$

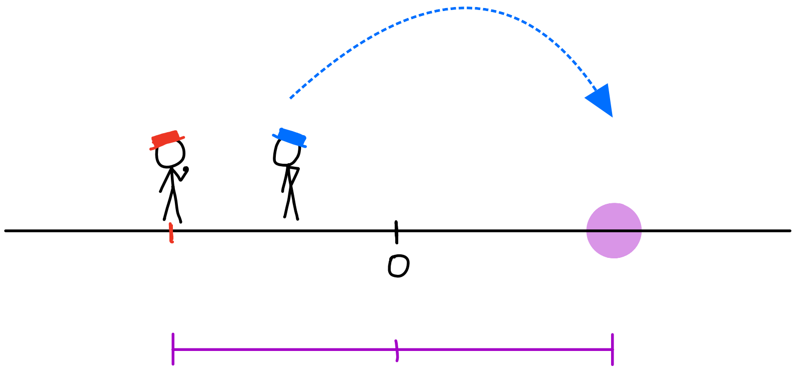
  - Repelling rule:

$$\mu_i(t+1) = \sum_{i\text{'s friends } j} w_{ij}\mu_j(t) + \sum_{i\text{'s opps } j} w_{ij}(\mu_i(t) - \mu_j(t))$$



Background: Opposing vs. Repelling

- Opposing rule:
  $\mu_i(t+1) = \sum_{i\text{'s friends } j} w_{ij}\mu_j(t) + \sum_{i\text{'s opps } j} (-1) w_{ij}\mu_j(t)$



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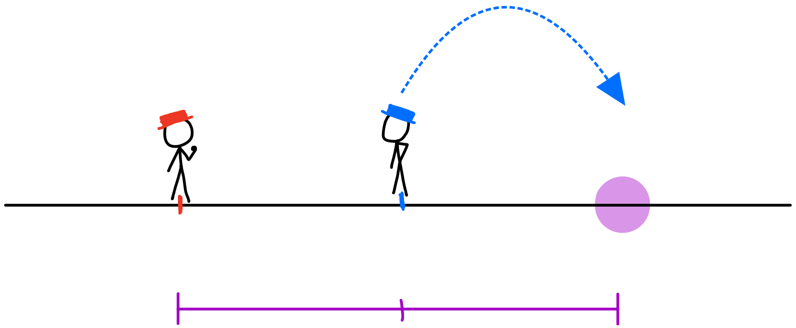
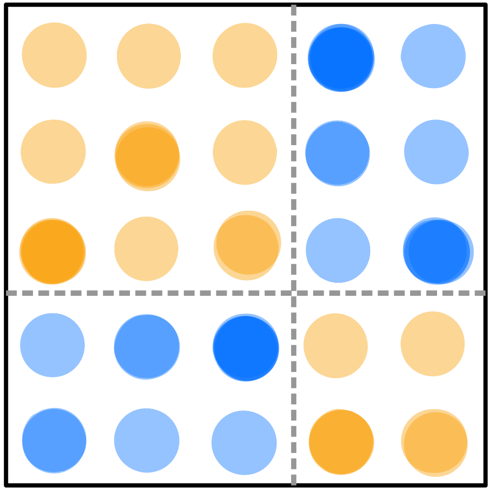
  $\mu_i(t+1) = \sum_{i\text{'s friends } j} w_{ij}\mu_j(t) + \sum_{i\text{'s opps } j} w_{ij}(\mu_i(t) - \mu_j(t))$

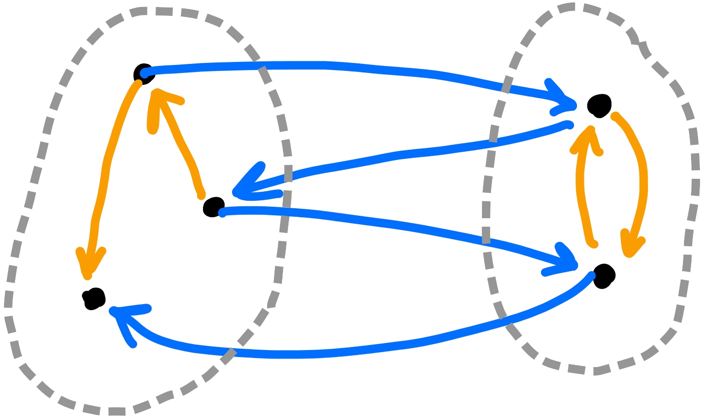


Background: Structural Balance (SB)

- Intuitively: "enemy of my enemy is my friend"
- Formally: weight product of every cycle is nonnegative



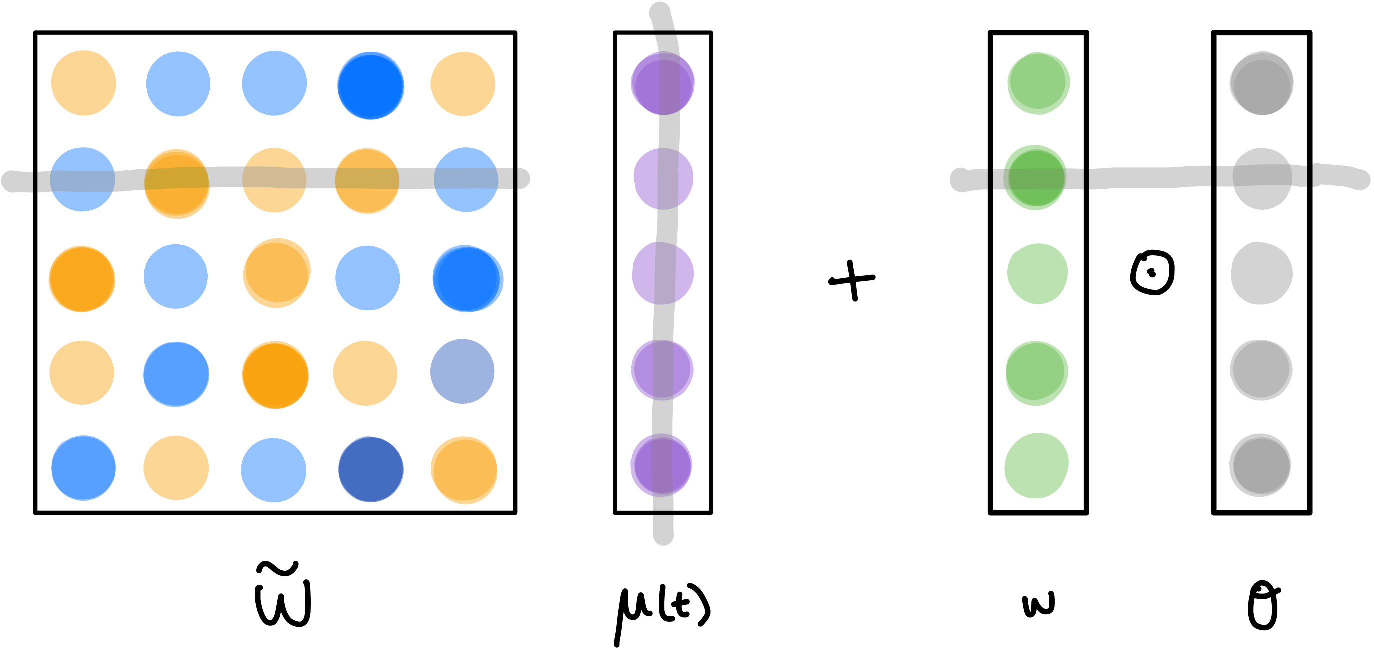
$\tilde{W}$ is SB



Background: Information

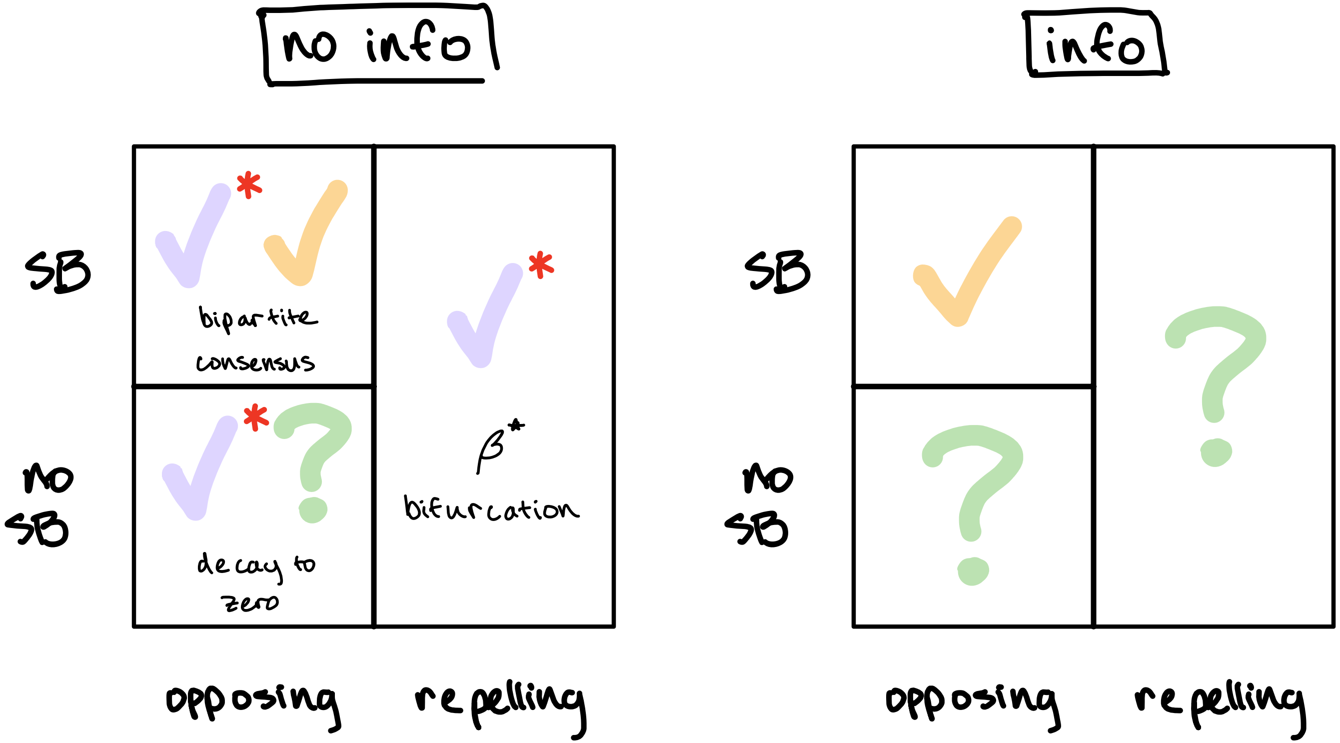
- Idea: your opinion is also influenced by external sources of info (eg. media), not just your neighbors
- Formally:

$$\mu(t+1) = \tilde{W}\mu(t) + w \odot \theta(t)$$

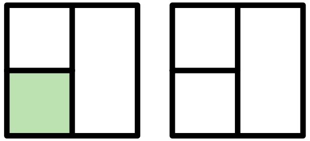


Existing Literature: Goal

- Shi et. al (2019)
- Lena, Merlino, Zenou ("LMZ") (2023)



Regime 1: (Non-Info) Opposing + No SB



Lemma

Suppose $\tilde{W} \in \mathbb{R}^{n \times n}$ and let $W := |\tilde{W}|$ denote the element-wise absolute value of $\tilde{W}$. Suppose further that W is row-stochastic and irreducible. If $\tilde{W}$ is not structurally balanced, then no eigenvalue of $\tilde{W}$ has magnitude exactly equal to 1.

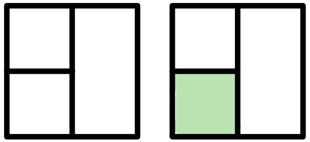
Pf Sketch.

Pf by contradiction. Spam triangle inequality, Perron-Frobenius on W to force SB. $\square$

Corollary

Under opposing dynamics (w/out info), opinions in a structurally unbalanced network will decay to 0.

Regime 2: (Constant Info) Opposing + No SB



- LMZ model:

$$\mu(t+1) = \tilde{W}\mu(t) + (c \odot \theta),$$

  where $c_i > 0$ for all i and θ is the "media sentiment" vector
- Results:
  - Common truth: if $\theta = \theta^* \mathbf{1}$, then the opinion vector converges to

$$\mu = \tilde{b} \cdot \theta^*,$$

    where $\tilde{b}$ is the vector of Katz-Bonacich centralities
  - Biased information: if $\theta_i = \theta_j$ for friends i, j and $\theta_i \neq \theta_j$ for opps i, j, then the opinion vector converges to

$$\mu = \tilde{b} \odot \theta$$

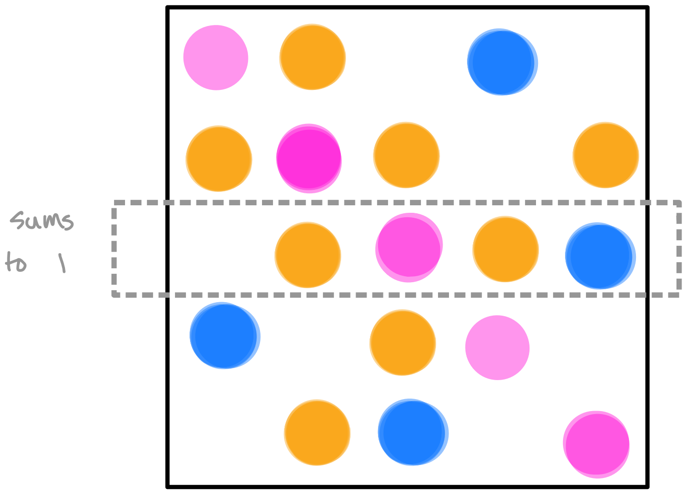
Regime 2: (Info) Opposing + No SB

- LMZ results hold without structural balance due to their assumption that $c_i > 0$ for all i (forces $\rho(\tilde{W}) < 1$)

Regime 3: (Constant Info) Repelling

- Repelling model from Shi:

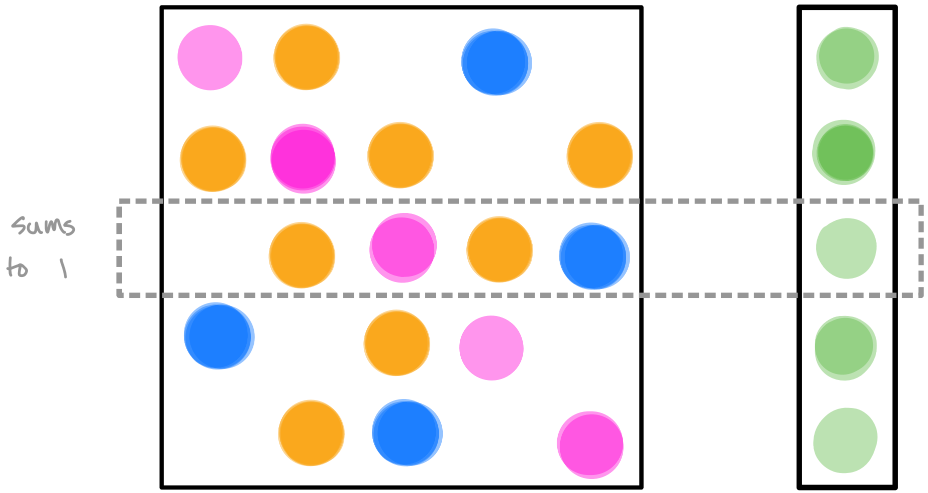
$$\mu(t+1) = M(\beta)\mu(t)$$



- My model:

$$\mu(t+1) = M'(\beta)\mu(t) + c \odot \theta,$$

where each $c_i \geq 0$ and
$c_i + \sum_j M'(\beta)_{ij} = 1$



Regime 3: (Constant Info) Repelling

Theorem

Let $G^+ = (V, E^+)$ and $G^- = (V, E^-)$ be undirected graphs on n nodes with (unsigned) Laplacians $L_+ \succeq 0$ and $L_- \succeq 0$. Fix $\alpha \geq 0, \beta \in \mathbb{R}$, a media-weight vector $c \in \mathbb{R}^n$ with $c_i \geq 0$, and let $D := \mathrm{diag}(c)$. Let $\theta \in \mathbb{R}^n$ be a constant "media sentiment" vector and define the constant input

$$c \circ \theta \equiv D\theta.$$

Define the symmetric update matrix

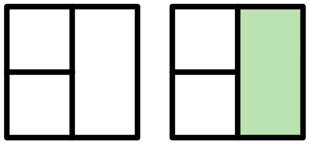
$$M'(\beta) := \overbrace{\boxed{I - \alpha L_+ - \beta L_-}}^{\text{Shi's model}} \overbrace{\boxed{- D}}^{\text{info leakage}}.$$

Consider the affine dynamical system

$$\mu(t+1) = M'(\beta)\mu(t) + c \odot \theta, \quad t = 0, 1, 2, \ldots$$

with initial condition $\mu(0) \in \mathbb{R}^n$.

Regime 3: (Constant Info) Repelling



Theorem (cont.)

Define $S := 2I - \alpha L_+ - D$. Then $\lambda_{\min}(M'(\beta)) > -1$ iff $S - \beta L_- \succ 0$. Assume S is positive definite on $\ker(L_-)$. Then the maximal $\underset{\text{antagonism level}}{\boxed{\beta}}$ for which $\lambda_{\min}(M'(\beta)) > -1$ is

$$\underset{\text{stability threshold}}{\boxed{\beta_{flip}(c)}} := \overset{\text{weakest link}}{\boxed{\inf}}_{\substack{x \neq 0 \\ x^\top L_- x > 0}} \frac{\overset{\text{stabilizing energy}}{\boxed{x^\top S x}}}{\underset{\text{antagonistic repulsion energy}}{\boxed{x^\top L_- x}}} = \inf_{\substack{x \neq 0 \\ x^\top L_- x > 0}} \frac{x^\top (2I - \alpha L_+ - D)\, x}{x^\top L_- x}.$$

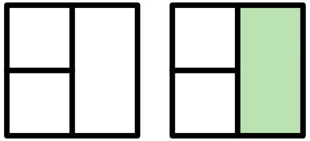
Hence,

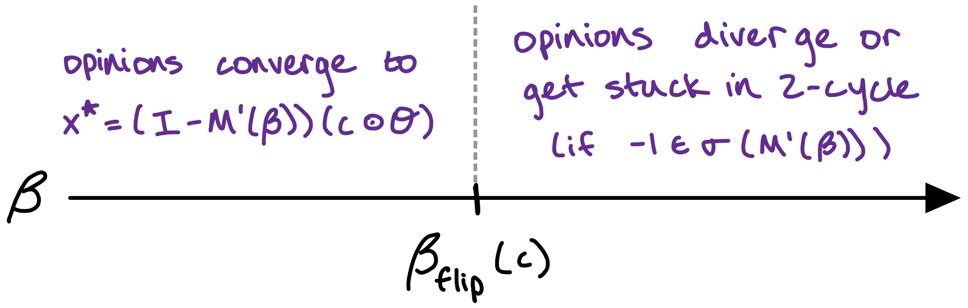
- *if $\beta < \beta_{flip}(c)$, we have $\lambda_{\min}(M'(\beta)) > -1$ and the system converges to the unique fixed point*

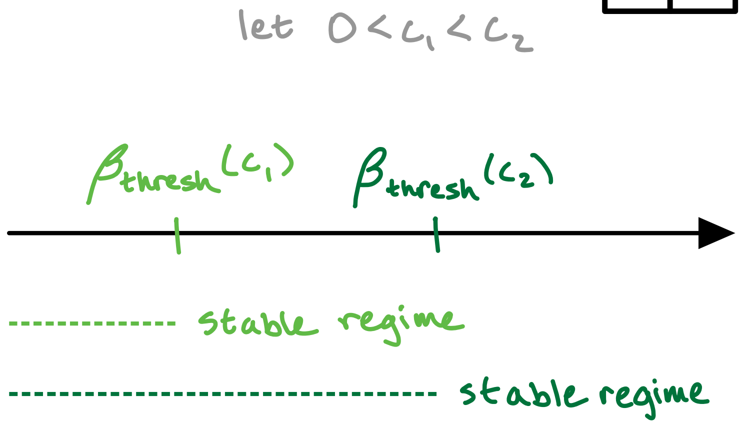
$$\mu^* = (I - M'(\beta))^{-1}(c \odot \theta)$$

- *if $\beta > \beta_{flip}(c)$, we have $\lambda_{\min}(M'(\beta)) \le -1$ and the system diverges*

Regime 3: (Constant Info) Repelling





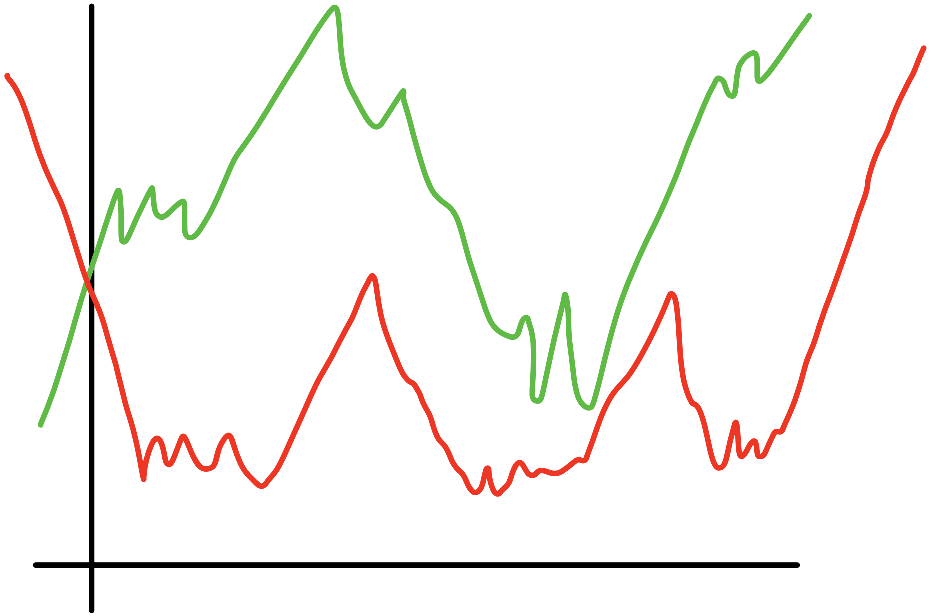


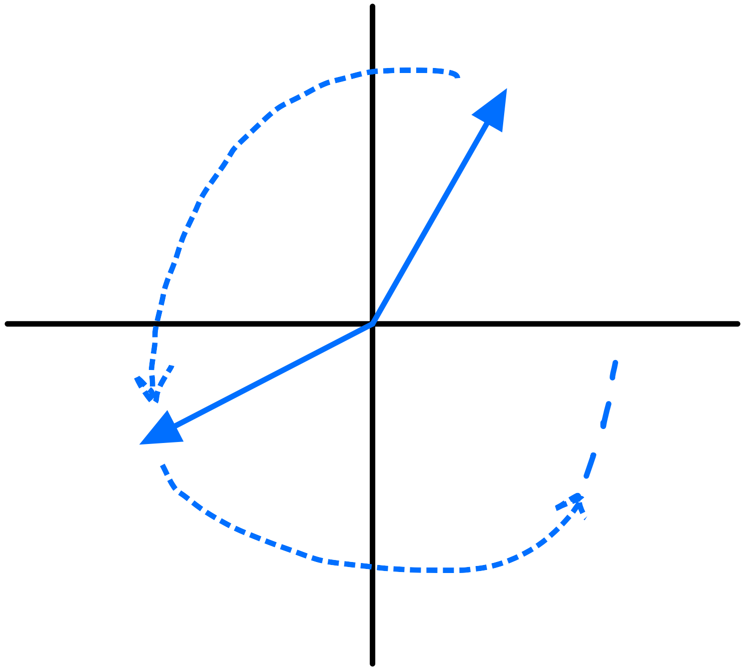
Pf. Sketch.

- show monotonicity of eigenvalues (in β) and harmless lower edge under positive self-weight
- exploit the symmetry of the matrix to orthogonally diagonalize $M'(\beta)$
- use orthogonal diagonalization to decompose into independent linear pieces (for each eigenvalue λ_i)
- analyze stability in terms of β and c via Weyl monotonicity + resolvent identities
- compute fixed point (in stable regime) via standard algebra

Oscillating information?

- Constant information seems like a crude assumption... what if the external signal is oscillating?

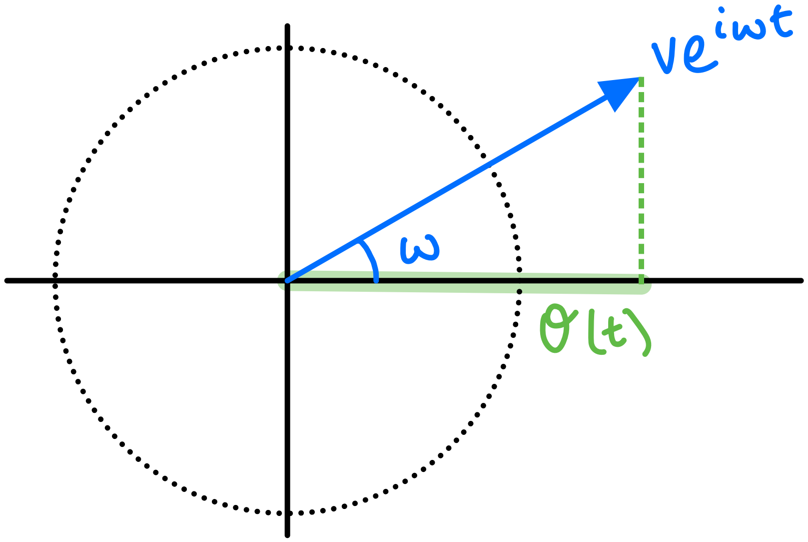




Regime 4: (Oscillating Info) Repelling

- Model: $x(t+1) = M'(\beta)x(t) + D\theta(t)$
- Do the "signal backend" in complex space: fix frequency $\omega \in [0, \pi]$ and let $v \in \mathbb{C}^n$ be a complex amplitude vector

$$\theta(t) = \Re\left(ve^{i\omega t}\right)$$



Regime 4: (Oscillating Info) Repelling

- Steady state?
  - Assuming $\rho(M'(\beta)) < 1$, there is a unique single-frequency periodic response of the form
  $$x^{\text{per}}(t) = \Re\big(H_\beta(\omega) v\, e^{i\omega(t-1)}\big),$$
  where
  $$H_\beta(\omega) := (I - e^{-i\omega} M'(\beta))^{-1} D$$
  - Moreover, every solution converges to $x^{\text{per}}(t)$ as $t \to \infty$
  - Pf Sketch: diff eq trick of finding a particular solution, then show the error goes to zero
- Amplification?
  - The gain along an eigenmode is given by:

$$g_k(\omega, \beta) := \frac{|a_k|}{|b_k|} = \frac{1}{|1 - e^{-i\omega}\lambda_k(\beta)|} = \frac{1}{\sqrt{1 + \lambda_k(\beta)^2 - 2\lambda_k(\beta)\cos\omega}}$$

Regime 4: (Oscillating Info) Repelling

- Effect of Antagonism?
  - Recall gain formula:

$$g_k(\omega, \beta) = \frac{1}{\sqrt{1 + \lambda_k(\beta)^2 - 2\lambda_k(\beta)\cos\omega}}$$

  - Gain maximized when $\lambda_k = \cos\omega$

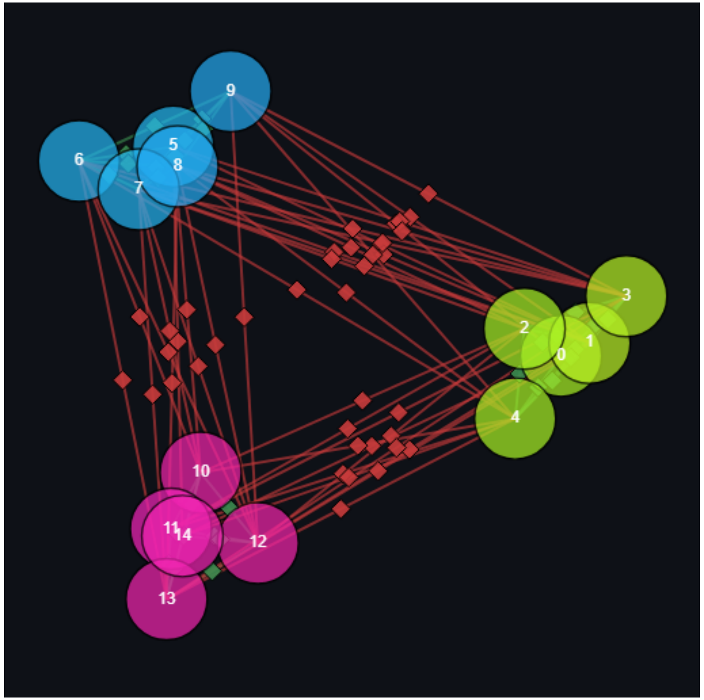
	Constant signals $(\omega = 0)$	**Alternating signals** $(\omega = \pi)$
Low antagonism (β small)	Weaker amplification	Stronger amplification
High antagonism (β near $\beta_{thresh}(c)$)	Stronger amplification	Weaker amplification

Optimizing Phase Lags

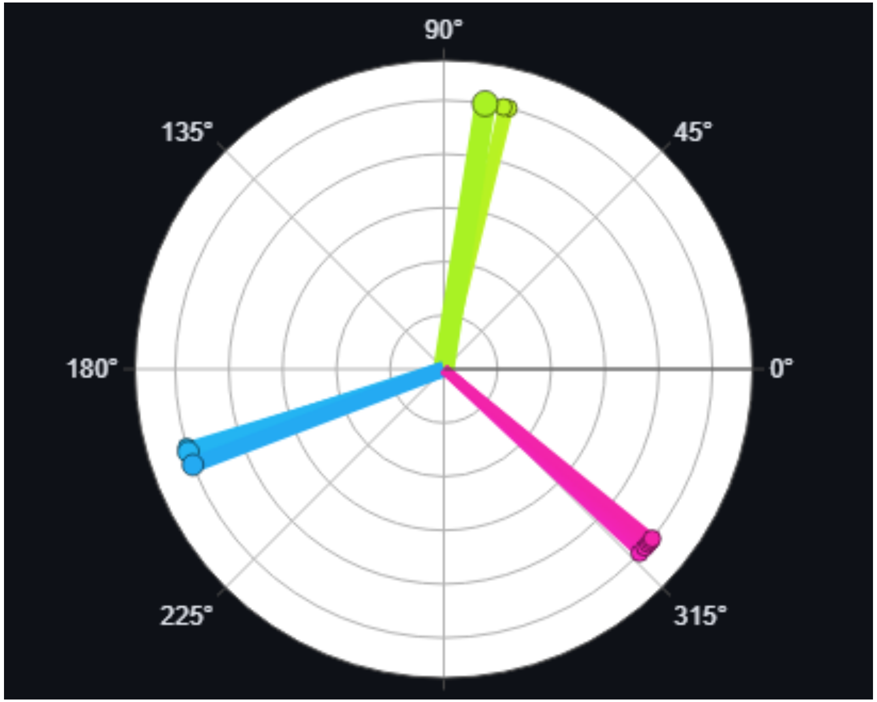
- What if we wanted to maximize disagreement by choosing when people receive information?
- Model:
$$a_i = r_i e^{i\phi_i}, \qquad r_i \geq 0, \qquad \phi_i \in [0, 2\pi),$$
where r_i is the magnitude of the signal received by agent i, while ϕ_i is the phase lag
- Budget: ℓ_∞ norm
- Objective: maximize variance of steady-state

Check out the webapp at https://spi-net.streamlit.app/

Optimizing Phase Lags



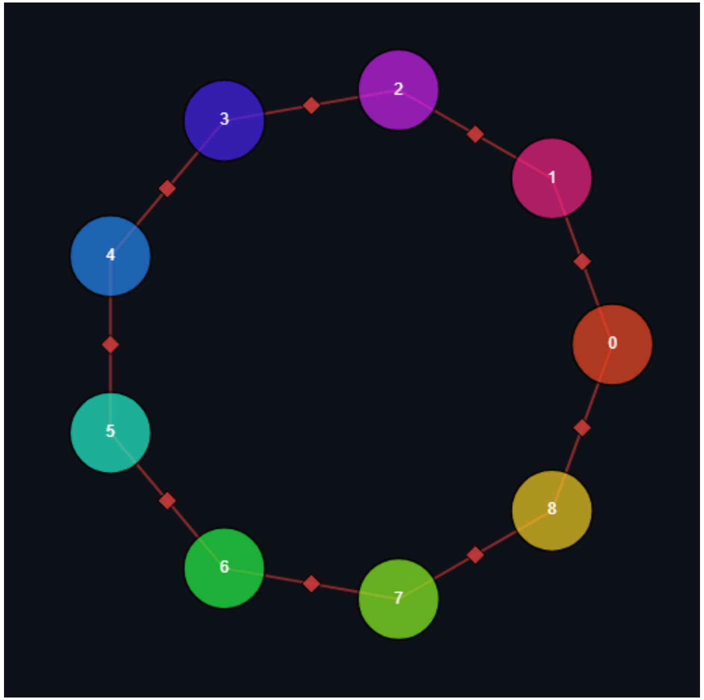
(a) Signed 3-community SBM.



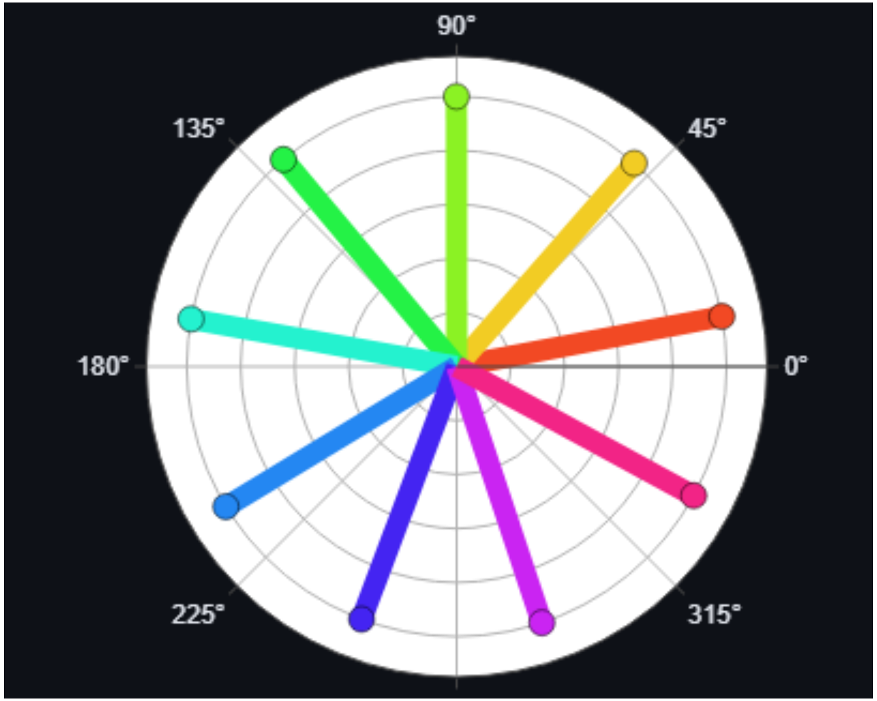
(b) Optimized 3-community SBM lags.

Figure: Three-block signed SBM: variance maximization under $\|r\|_\infty \leq 1$ gives a three-spoke phase pattern.

Optimizing Phase Lags



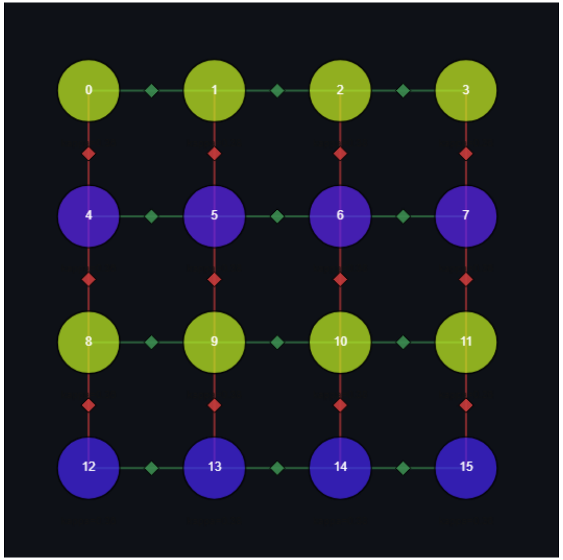
(a) Antagonistic 9-cycle.



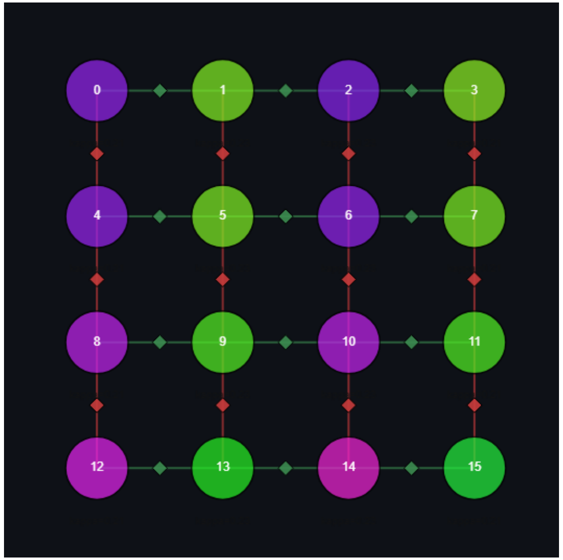
(b) Optimized 9-cycle lags.

Figure: Antagonistic 9-cycle: variance maximization under $\|r\|_\infty \leq 1$ gives an (evenly spaced) 9-spoke phase pattern.

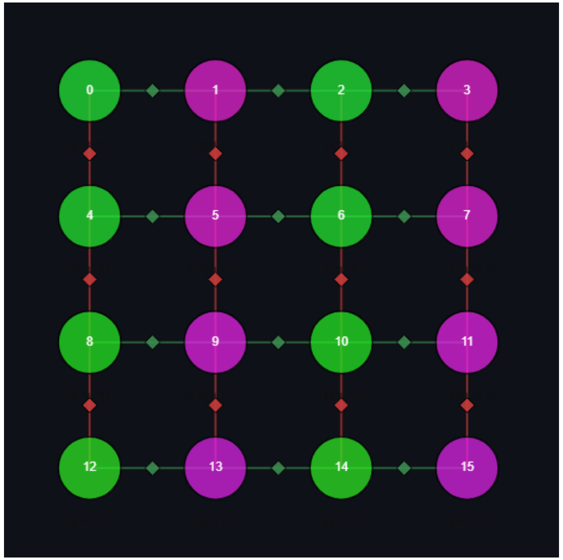
Optimizing Phase Lags



(a) Phase lag pattern on a lattice at frequency $\omega = 0.48$.



(b) Phase lag pattern on a lattice at frequency $\omega = 0.84$.

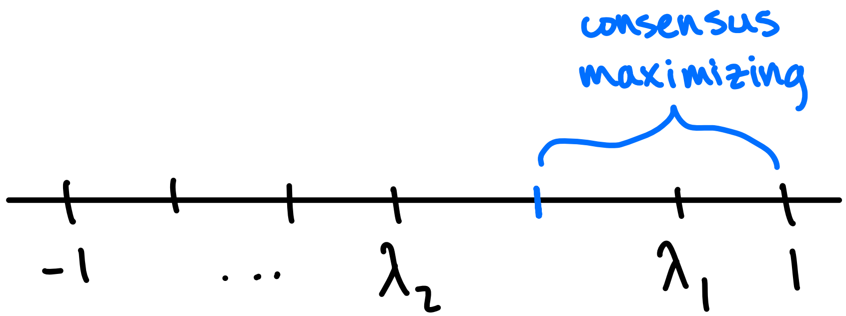


(c) Phase lag pattern on a lattice at frequency $\omega = 1.32$.

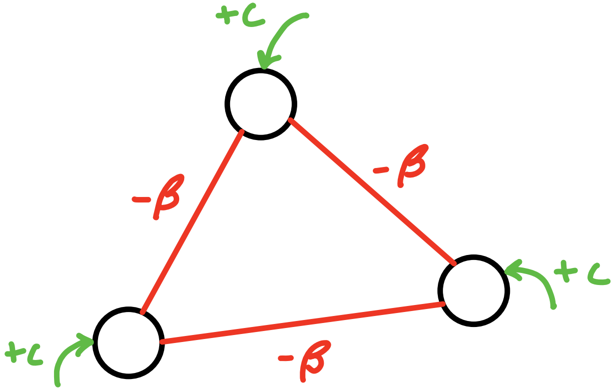
Figure: Lattice network: variance maximization under $\|r\|_\infty \leq 1$ give qualitatively different phase lag profiles at different frequencies.

Signed vs. Unsigned Networks

- Prove a "Perron barrier to disagreement" for unsigned networks: if
$$\cos\omega \geq \boxed{\frac{\lambda_1 + \lambda_2}{2}},$$
then the unique largest modal gain is attained at $k = 1$ (consensus-like)



- Prove **this barrier can fail** for signed networks!
  - Pf Sketch: frustrated triangle, show $\lambda_{\text{cons}} < \lambda_{\text{dis}} < 1$, show gain increasing in λ at $\omega = 0$, by continuity there exists ϵ s.t. for all $\omega \in (0, \epsilon)$, gain is maximized along disagreement subspace



Open Questions

- Can we say something sharper about what signed networks offer?
- Sequence space models?
- Theory of how to optimize oscillating signals + phase lag for maximum amplification?

Thank You!

- Shoutout Prof. Ben Golub for his evidently infinite patience shepherding my often incoherent ramblings along more sensible eigenmodes